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Internal gravity waves in shear flows at large Reynolds number

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The effects of viscosity and heat conduction on the propagation of internal gravity waves are examined. These waves propagate in a stably stratified, parallel shear flow with one critical level. The Boussinesq approximation is adopted. For large Reynolds number the governing sixth-order differential equation is solved by analytical methods. In the limit of large Reynolds number it is found that the reflection and transmission coefficients for a wave incident in a viscous fluid are the same as in the inviscid case. Hence over-reflection can also occur in a viscous fluid. For the perturbed velocity components at the critical level, asymptotic expressions are derived. The results we obtain are valid for smooth, but otherwise arbitrary, shear-flow and density profiles.

1. Introduction

In the inviscid theory of linear internal gravity waves the differential equation for the vertical velocity is of second order. The presence of a stratified, parallel shear flow causes this equation to possess a singularity at any critical level, i.e. at any level where the shear-flow velocity is equal to the horizontal phase velocity of the wave. At this level the Reynolds stress is discontinuous (Eliassen & Palm 1961). Another important aspect of the inviscid theory is the occurrence of over-reflection, which means that the amplitude of the reflected wave exceeds that of the incident one (see, e.g., Jones 1968; McKenzie 1972; Eltayeb & McKenzie 1975; and Van Duin & Kelder 1982). A necessary condition for the occurrence of this phenomenon is that the Richardson number at the critical level is sufficiently small. For an ample discussion on over-reflection the reader is referred to Acheson (1976).

It should be noted that nonlinearity may become important in the vicinity of the critical level. Brown & Stewartson (1980, 1982*a*, *b*) examined the nonlinear interaction of the internal gravity wave with its critical level for large values of a representative Richardson number. They have shown that, however small the amplitude of the forced wave may be, the linear steady model breaks down on a timescale that is inversely proportional to the amplitude of the wave. On a larger timescale the reflection and transmission coefficients change. These authors excluded the effects of viscosity and heat conduction. The mechanism of the combined effects of dissipation and nonlinearity is not yet well understood.

It is the aim of this paper to examine the effects of viscosity and heat conduction on the propagation of linear internal gravity waves. When these effects are taken into account, the governing differential equation is of sixth order (Koppel 1964; Hazel 1967; Baldwin & Roberts 1970). The Reynolds stress is now a continuous function of height because the singularity at the critical level is removed. In the limit of large

Reynolds number various aspects of the inviscid theory are recovered, including the possible occurrence of over-reflection. The Boussinesq approximation is adopted and we will be concerned with the presence of one critical level.

The situation as described by this steady model will persist for a certain period of time (when the effect of the transient phenomenon has been damped out) until the nonlinear terms ultimately become important, and it is expected that viscosity and heat conduction will delay the moment at which these terms can no longer be neglected. It is of interest to refer to a paper by Tung, Ko & Chang (1981). These authors derived a measure of the relative importance of viscosity as compared with nonlinear effects. They found that it can be characterized by a single parameter. Large values of this parameter indicate that viscosity dominates nonlinearity.

It is assumed that the dissipation results from molecular viscosity and heat conduction. For the thermosphere this appears to be a reasonable approximation (see, e.g. Gossard & Hooke 1975; Fritts & Geller 1976; Gill 1982). For waves with a small amplitude, Fritts & Geller (1976) found a stabilizing effect of molecular dissipation near the critical level. When the amplitude of the incident wave is sufficiently large, however, convective instability can be induced (Fritts & Geller 1976; Fritts 1982). Another effect limiting the wave amplitude is the transfer of energy to other scales by parametric instability (see, e.g. Mied 1976; Drazin 1977; Grimshaw 1980). For the latter mechanism the growth rates of the unstable modes increase with the amplitude of the incident wave.

In §2 the problem is posed. In §3 the method of matched asymptotic expansions is used to find the common region of validity of the so-called outer and inner solutions of the governing equation. This method also allows the construction of a uniformly valid approximation to the solution of this equation (§4). Results based on the analytical solution of the governing equation are also discussed in that section. It is remarked that extensive use is made of a paper by Baldwin & Roberts (1970).

2. The problem

We consider a viscous and heat conducting fluid with a parallel shear flow $\mathbf{U} = U(y)\mathbf{i}$, where $\mathbf{i}$ is the unit vector parallel to the x-axis of a Cartesian coordinate system. The y-axis is taken to be parallel to the vertical direction and increasing y corresponds to increasing height. The fluid is stratified and incompressible. The undisturbed density ρ_0 depends on y only and the stratification is stable, i.e. $\mathrm{d}\rho_0/\mathrm{d}y < 0$. The Brunt–Väisälä frequency N is defined by $N^2(y) = -g\rho_0^{-1}(\mathrm{d}\rho_0/\mathrm{d}y)$, where g is the gravitational acceleration. We further adopt the Boussinesq approximation. The perturbation quantities have a dependence on x and t of the form $\exp\{\mathrm{i}\alpha(x-ct)\}$, where α and c are the horizontal wavenumber and the horizontal phase velocity, respectively. Only real values of c are considered. The equation for the vertical component ϕ of the velocity (Baldwin & Roberts 1970) reads

$$\left[\mathrm{D}^2-\alpha^2-\frac{\mathrm{i}\alpha}{\chi}(U-c)\right]\left[\left\{\mathrm{D}^2-\alpha^2-\frac{\mathrm{i}\alpha}{\nu}(U-c)\right\}(\mathrm{D}^2-\alpha^2)\phi+\frac{\mathrm{i}\alpha U''}{\nu}\phi\right]=\frac{N^2\alpha^2}{\nu\chi}\phi, \quad (2.1)$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}y$, ν is the kinematic viscosity and χ the thermal conductivity. The prime has the same meaning as D.

It is assumed that c lies in the range of U and that there is only one point, $y = y_c$, where $U = c$. Furthermore, the slope of the shear flow at $y = y_c$ is taken positive. The point $y = y_c$ is called the critical point. The name critical level is also common. We consider smooth but otherwise arbitrary velocity and density profiles $U(y)$ and $\rho_0(y)$.

The Richardson number at the critical level is defined by

$$J = \left(\frac{N_c}{U'_c}\right)^2, \tag{2.2}$$

where the subscript c means evaluation at $y = y_c$.

Introducing L and V as the scales of length and velocity, (2.1) can be written in the dimensionless form

$$[\mathrm{D}^2-\alpha^2-\mathrm{i}\alpha RP(U-c)][\{\mathrm{D}^2-\alpha^2-\mathrm{i}\alpha R(U-c)\}(\mathrm{D}^2-\alpha^2)\phi+\mathrm{i}\alpha RU''\phi] = (\alpha^2R^2PRi)\phi, \tag{2.3}$$

where
$$R = \frac{VL}{\nu}, \quad P = \frac{\nu}{\chi}, \quad Ri = \left(\frac{LN}{V}\right)^2. \tag{2.4}$$

Here R and P denote the Reynolds and Prandtl numbers. The overall Richardson number Ri depends on y because the Brunt–Väisälä frequency N depends on this coordinate.

In the formal limit as $R \rightarrow \infty$, with fixed $y \neq y_c$, (2.3) reduces to the Taylor–Goldstein equation

$$\phi'' + \left\{\frac{Ri}{(U-c)^2} - \frac{U''}{U-c} - \alpha^2\right\}\phi = 0. \tag{2.5}$$

It is assumed that the Reynolds number R is so large that, near the critical level, viscosity and heat conduction are only important in a narrow region near this level. In the outer regions, i.e. the regions away from the critical level, (2.3) may then be approximated by (2.5), because in these regions the fluid may be considered as inviscid. To find the connection between the solutions of (2.5) in the disjoint outer regions, the method of matched asymptotic expansions will be used. This method also allows the construction of a uniformly valid approximation to the solution of (2.3). The results to be derived are valid in the limit of vanishing viscosity and heat conduction ($R \rightarrow \infty$, P fixed).

We note in passing that in the derivation of (2.1) it is supposed that all undisturbed quantities vary with height only. As Jones (1985) remarked, the undisturbed temperature should then be a linear function of height to satisfy the equation of heat transfer. Likewise, the shear flow should then vary linearly with height to satisfy the Navier–Stokes equation. In the limit of vanishing viscosity and heat conduction, however, the undisturbed quantities necessarily depend on height only. Moreover, $U(y)$ and $N(y)$ may be arbitrary functions of height in this limit.

3. The outer and inner solutions, matching

For an inviscid, Boussinesq fluid the Taylor–Goldstein equation (2.5) is the model equation for the propagation of internal gravity waves. The solution of this 'inviscid' equation, which will be called the inviscid solution (denoted by $\phi_{\mathrm{inv}}(y)$), is of the form

$$\phi_{\mathrm{inv}}(y) = A\phi_1(y) + B\phi_2(y), \tag{3.1}$$

where $\phi_1(y)$ and $\phi_2(y)$ are the Frobenius solutions about $y = y_c$:

$$\phi_1(y) = (y-y_c)^{\frac{1}{2}+\mu}\left\{1 + \sum_{n=1}^{\infty} a_n(y-y_c)^n\right\}, \tag{3.2a}$$

$$\phi_2(y) = (y-y_c)^{\frac{1}{2}-\mu}\left\{1 + \sum_{n=1}^{\infty} b_n(y-y_c)^n\right\}. \tag{3.2b}$$

A and B are constants and

$$\mu = (\tfrac{1}{4}-J)^{\frac{1}{2}}. \tag{3.3}$$

It is remarked that for $J = \frac{1}{4}$ one of the Frobenius solutions of (2.5) has a logarithmic singularity at $y = y_c$. This special value of J will be excluded because the results to be derived can easily be extended to the case $J = \frac{1}{4}$, without recourse to the logarithmic solution.

To make the solutions (3.2*a*, *b*) one-valued, one has to introduce a branch cut in the complex y-plane. This cut should be made upwards, hence $\arg(y-y_c) = -\pi$ for $y < y_c$. This choice has been motivated by Booker & Bretherton (1967) and by Baldwin & Roberts (1970).

When viscosity and heat conduction are included, the Taylor–Goldstein equation (2.5) is the limit equation of (2.3), i.e. for fixed $y \neq y_c$, (2.3) reduces to (2.5) as $R \to \infty$. The solution of (2.5) is called the outer solution in this case and will be denoted by $\phi_o(y)$ instead of $\phi_{inv}(y)$. When we substitute the asymptotic expansion

$$\phi(y) = \phi^{(0)}(y) + R^{-1}\phi^{(1)}(y) + R^{-2}\phi^{(2)}(y) + \dots, \tag{3.4}$$

into (2.3), it is found that $\phi^{(0)}(y)$ is a solution of (2.5). Thus the outer solution $\phi_o(y)$ is the first term in the expansion (3.4). Hence, for fixed $y \neq y_c$ it is the leading-order approximation to $\phi(y)$.

In the outer regions $y < y_c$ and $y > y_c$, $\phi_o(y)$ is a linear combination of the functions (3.2*a*, *b*). Since the outer regions are separated by a thin boundary layer or viscous layer, however, one might expect that the outer solution takes the form

$$\phi_o(y) = \begin{cases} A^+\phi_1(y) + B^+\phi_2(y) & (y > y_c), & (3.5a) \\ A^-\phi_1(y) + B^-\phi_2(y) & (y < y_c), & (3.5b) \end{cases}$$

instead of the form (3.1) for the inviscid solution. $A^{\pm}$ and $B^{\pm}$ are constants. For later use the leading-order behaviour of the outer solution near the critical level is given, namely

$$\phi_o(y) \sim \begin{cases} A^+(y-y_c)^{\frac{1}{2}+\mu} + B^+(y-y_c)^{\frac{1}{2}-\mu} \quad \text{as } y \downarrow y_c, & (3.6a) \\ -\mathrm{i}A^-\mathrm{e}^{-\mathrm{i}\mu\pi}(y_c-y)^{\frac{1}{2}+\mu} - \mathrm{i}B^-\mathrm{e}^{\mathrm{i}\mu\pi}(y_c-y)^{\frac{1}{2}-\mu} \quad \text{as } y \uparrow y_c. & (3.6b) \end{cases}$$

Substituting the new variable,

$$\eta = (\mathrm{i}\alpha R U_c')^{\frac{1}{3}}(y-y_c), \tag{3.7}$$

into (2.3), it is found that in the formal limit as $R \to \infty$, with η fixed, the resulting equation (with η as the independent variable) reduces to the limit equation

$$\left(P^{-1}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - \eta\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - \eta\right)\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} + J\phi = 0. \tag{3.8}$$

Apparently the thickness of the boundary layer is of order $R^{-\frac{1}{3}}$. In this layer the solution of (3.8), known as the (leading-order) inner solution, is valid as an approximation to the solution of (2.3). It should be noted that the solution of (3.8) is the first term in an asymptotic expansion of the form (3.4), with y and R replaced by η and $R^{\frac{1}{3}}$, respectively.

To obtain a single uniform approximation, the inner solution should be matched with the outer solution. Then the ratios A^+/A^- and B^+/B^- in (3.5) are determined and, consequently, the connection between the solutions in the disjoint outer regions is made.

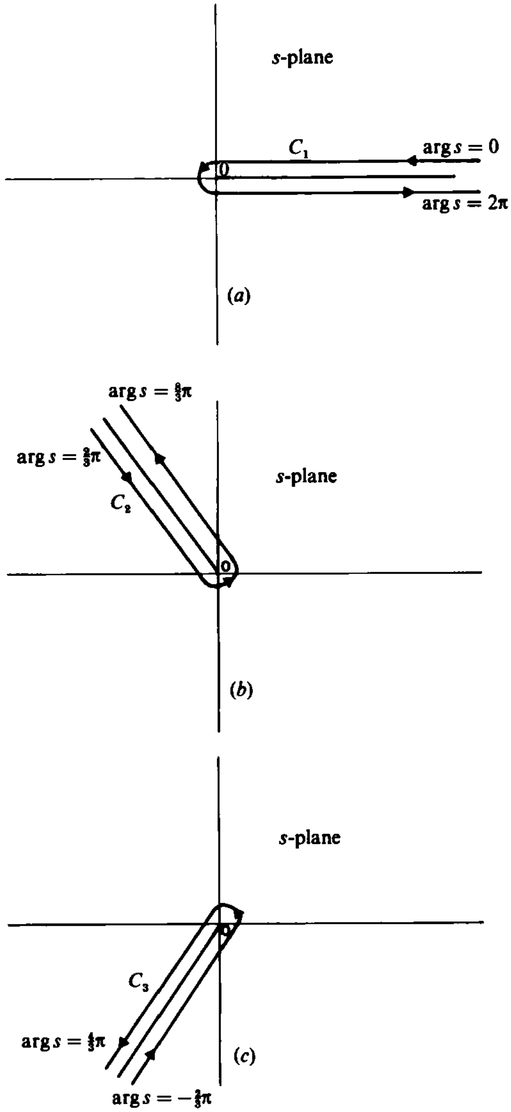

FIGURE 1. Contours C_j for the functions $u_3(\eta)$ and $v_3(\eta)$ as defined by (3.9). (After Baldwin & Roberts 1970.)

3.1. *Matching*

Baldwin & Roberts (1970; hereinafter referred to as BR), constructed a base of solutions of (3.8). For $P \neq 1$, three of these solutions are given by (see the formula (4.3) in their paper),

$$u_j(\eta) = \int_{C_j} e^{\eta s - \frac{1}{3}s^3} f(s)\, s^{\mu - \frac{3}{2}}\, ds \quad (j = 1, 2, 3), \tag{3.9}$$

where
$$f(s) = {}_1F_1(\tfrac{1}{6} + \tfrac{1}{3}\mu; 1 + \tfrac{2}{3}\mu; \tfrac{1}{3}(1 - P^{-1})\, s^3), \tag{3.10}$$

is the confluent hypergeometric function and

$$\mu = \begin{cases} (\frac{1}{4} - J)^{\frac{1}{2}}, & (J < \frac{1}{4}), \\ i\sigma, \quad \sigma = (J - \frac{1}{4})^{\frac{1}{2}} & (J > \frac{1}{4}). \end{cases} \tag{3.11}$$

The contours C_j are sketched in figure 1. The remaining three solutions, denoted by $v_j(\eta)$, are obtained by replacing μ by $-\mu$. It is remarked that BR use the symbols z and m instead of η and μ, where $\mu = 3m$. The inner solution, denoted by $\phi_i(\eta)$, is of the form

$$\phi_i(\eta) = \sum_{j=1}^{3} \{\alpha_j u_j(\eta) + \beta_j v_j(\eta)\}, \tag{3.12}$$

where α_j and β_j are constants to be determined by matching.

We now introduce the so-called intermediate variable (Kevorkian & Cole 1981 chapter 2)

$$\zeta = \frac{y - y_c}{\epsilon(R)}, \tag{3.13}$$

for real and positive functions $\epsilon(R)$ that are contained in the class

$$R^{-\frac{1}{3}} = o(\epsilon(R)), \epsilon(R) = o(1) \quad \text{as } R \to \infty. \tag{3.14a, b}$$

With the aid of (3.13) and (3.14) the size of the overlap domain (i.e. a common region of validity of the inner and outer solutions) can be found from the matching principle

$$\phi_i(\eta) \sim \{\phi_0(y) \sim\} \quad \text{as } R \to \infty, \zeta \text{ fixed}, \tag{3.15}$$

where $\{\phi_0(y) \sim\}$ is the asymptotic behaviour of $\phi_0(y) = \phi_0(y_c + \epsilon\zeta)$ as $R \to \infty$ for fixed ζ. For $\zeta > 0$ (or $\zeta < 0$) this corresponds to the behaviour of $\phi_0(y)$ *as* $y \downarrow y_c$ (or $y \uparrow y_c$), cf. (3.13) and (3.14*b*). Hence (3.15) implies that, for fixed ζ, and as $R \to \infty$,

$$\phi_i(\eta) \sim \begin{cases} A^+(y-y_c)^{\frac{1}{2}+\mu} + B^+(y-y_c)^{\frac{1}{2}-\mu} & (\zeta > 0), \quad (3.16a) \\ -iA^- e^{-i\mu\pi}(y_c - y)^{\frac{1}{2}+\mu} - iB^- e^{i\mu\pi}(y_c-y)^{\frac{1}{2}-\mu} & (\zeta < 0), \quad (3.16b) \end{cases}$$

cf. (3.6). From (3.7) and (3.14*a*) it follows that $|\eta| \to \infty$ as $R \to \infty$. In other words, to match the inner solution (3.12) with the outer solution (3.5), the behaviour of (3.12) as $\eta \to \pm\infty \exp(\frac{1}{6}i\pi)$ has to be known. The factor $\exp(\frac{1}{6}i\pi)$ arises because, in view of (3.7), η is complex for real y. The asymptotic expressions for $u_j(\eta)$ and $v_j(\eta)$ are listed in tables 1 and 2 in BR. Only the sectors T_1 and T_2 in these tables are of interest here. It should be noted that these results are only valid for $P \neq 1$. The case $P = 1$ has been discussed by Koppel (1964) and by Gage & Reid (1968).

It turns out that only the solutions $u_3(\eta)$ and $v_3(\eta)$ can be matched with the outer solution, because two of the remaining solutions tend exponentially to infinity for $\zeta > 0$ (corresponding to $\text{Re}(\eta) \to \infty$), where Re denotes the real part, and the other two have this behaviour for $\zeta < 0$ ($\text{Re}(\eta) \to -\infty$). Therefore we must take $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ and, consequently, (3.12) reduces to

$$\phi_i(\eta) = \alpha_3 u_3(\eta) + \beta_3 v_3(\eta). \tag{3.17}$$

For $J < \frac{1}{4}$ the asymptotic behaviour of $u_3(\eta)$ as $R \to \infty$ (with ζ fixed) is of the form (BR)

$$u_3(\eta) \sim \begin{cases} \dfrac{2\pi i}{\Gamma(\frac{3}{2}-\mu)} (i\alpha R U_c')^{\frac{1}{3}(\frac{1}{2}-\mu)} (y-y_c)^{\frac{1}{2}-\mu} & (\zeta > 0), \quad (3.18a) \\[2ex] \dfrac{2\pi}{\Gamma(\frac{3}{2}-\mu)} e^{i\mu\pi} (i\alpha R U_c')^{\frac{1}{3}(\frac{1}{2}-\mu)} (y_c-y)^{\frac{1}{2}-\mu} & (\zeta < 0), \quad (3.18b) \end{cases}$$

where μ is given by (3.11). The asymptotic behaviour of $v_3(\eta)$ may be obtained by replacing μ by $-\mu$ in (3.18). For $J > \frac{1}{4}$, μ must be replaced by $i\sigma$, where σ is defined by (3.11).

Combining (3.16), (3.17) and (3.18) gives

$$A^- = A^+, \quad B^- = B^+, \tag{3.19}$$

$$\alpha_3 = \frac{B^+\Gamma(\frac{3}{2}-\mu)}{2\pi \mathrm{i}}(\mathrm{i}\alpha R U_c')^{\frac{1}{3}(\mu-\frac{1}{2})}, \tag{3.20a}$$

$$\beta_3 = \frac{A^+\Gamma(\frac{3}{2}+\mu)}{2\pi \mathrm{i}}(\mathrm{i}\alpha R U_c')^{-\frac{1}{3}(\mu+\frac{1}{2})}. \tag{3.20b}$$

The result (3.19) implies that the outer solution (3.5) reduces to

$$\phi_o(y) = A^+\phi_1(y) + B^+\phi_2(y). \tag{3.21}$$

It is therefore of the same form as the inviscid solution (3.1). When A^+ and B^+ are known, the coefficients α_3 and β_3 in the inner solution (3.17) are determined by (3.20).

Finally, it should be noted that (3.15) is an extension of Prandtl's matching principle (Nayfeh 1973). Both principles require that the behaviours of the inner solution as $\eta \to \pm\infty \exp(\mathrm{i}\pi/6)$ and the outer solution as $y \to y_c^{\pm}$ are in agreement. Using Prandtl's matching principle, however, it is not possible to find an overlap domain for matching. On the other hand, from (3.13)–(3.15) it follows that there exists such a domain. The size of it is found to be $R^{-\frac{1}{3}} \ll y - y_c \ll 1$ as $R \to \infty$, where the symbol $\ll$ is used instead of the small-o symbol.

4. Results

A uniformly valid leading-order approximation to the solution $\phi(y)$ of (2.3) is obtained by adding the inner and outer solutions (3.17) and (3.21) together and subtracting the common part. The common part (cp) consists of the common terms in (3.17) and (3.21) that cancel out in the matching. Thus, the uniformly valid first approximation is of the form

$$\phi_{\mathrm{un}}(y) = \phi_i(\eta) + \phi_o(y) - \mathrm{cp}. \tag{4.1}$$

The common part is given by

$$\mathrm{cp} = A^+(y-y_c)^{\frac{1}{2}+\mu} + B^+(y-y_c)^{\frac{1}{2}-\mu}. \tag{4.2}$$

The difference $\phi(y) - \phi_{\mathrm{un}}(y)$ is uniformly $o(1)$ as $R \to \infty$ on *every* closed interval $[y_1, y_2]$.

In a narrow region near the critical level the inner solution (3.17) is valid as an approximation to the solution $\phi(y)$ of (2.3). It is recalled that this approximation to the perturbed vertical velocity, obtained by leading-order matching, could only be constructed by discarding four of the six solutions in the general expression (3.12). Away from the critical level the outer solution (3.21) is valid as an approximation in the sense that, for every fixed $y \neq y_c$,

$$\phi(y) - \phi_o(y) = o(1) \quad \text{as } R \to \infty, \tag{4.3}$$

because the result

$$\frac{\alpha_3 u_3(\eta)}{B^+(y-y_c)^{\frac{1}{2}-\mu}} \sim 1, \quad \frac{\beta_3 v_3(\eta)}{A^+(y-y_c)^{\frac{1}{2}+\mu}} \sim 1 \quad \text{as } R \to \infty, \tag{4.4a, b}$$

(fixed $y \neq y_c$) implies that the inner solution minus the common part in the expansion (4.1) vanishes as $R \to \infty$.

4.1. *Uniform validity of the outer limit*

Hazel (1967) derived asymptotic expansions for the six solutions of (3.8), valid for $P \neq 1$ and $J > \frac{1}{4}$ (see his formulae (2.1)–(2.6)). Combining these results with those derived in BR gives the expressions

$$u_3(\eta) = \frac{2\pi i}{\Gamma(\frac{3}{2} - i\sigma)} \eta^{\frac{1}{2} - i\sigma} \{1 + O(\eta^{-3})\}, \tag{4.5a}$$

$$v_3(\eta) = \frac{2\pi i}{\Gamma(\frac{3}{2} + i\sigma)} \eta^{\frac{1}{2} + i\sigma} \{1 + O(\eta^{-3})\}, \tag{4.5b}$$

valid as $\eta \to \infty \exp(\frac{1}{6} i\pi)$; σ is defined by (3.11). From (3.7), (3.17), (3.20), (4.2) and (4.5) it then follows that

$$\phi_i(\eta) - cp = O(\eta^{-\frac{5}{2}}) \quad \text{as } \eta \to \infty \exp(\tfrac{1}{6} i\pi), \tag{4.6}$$

which implies that

$$\phi_i(\eta) - cp = o(1) \quad \text{as } R \to \infty, \tag{4.7}$$

uniformly for $y \geqslant y_1 > y_c$. The asymptotic relation,

$$\phi(y) - \phi_{un}(y) = o(1) \quad \text{as } R \to \infty, \tag{4.8}$$

is uniformly valid on every closed interval on the y-axis. Combining the results (4.1), (4.7) and (4.8) it is then found that (4.3) is uniformly valid for $y_1 \leqslant y \leqslant y_2$, where $y_1 > y_c$. In a similar manner it can be proven that this is also true for every closed interval $[y_1, y_2]$ *below* the critical level, namely $y_2 < y_c$. If the parameter $i\sigma$ in (4.5) is replaced by μ as give by (3.11), this result can be extended to be valid for all $J \neq \frac{1}{4}$.

In conclusion it is observed that for $P \neq 1$ and $J \neq \frac{1}{4}$, (4.3) is uniformly valid on every closed interval $[y_1, y_2]$ below or above the critical level. This result will be applied in the next subsection.

4.2. *Reflection and transmission coefficients*

Assuming that the Reynolds number is large, the effect of viscosity is only important in a narrow region near the critical level and in regions far away from this level. In the latter regions the effect of viscosity is cumulative, i.e. it causes the wave to vanish as $y \to \pm\infty$.

In the homogeneous regions $|y - y_c| \gg 1$ the imaginary part of the vertical wavenumber is of order R^{-1} as $R \to \infty$. Consequently, when a wave propagates from the region $1 \ll y_c - y \ll R$ below the critical level to the region $1 \ll y - y_c \ll R$ above this level or vice versa, the cumulative effect of viscosity on the amplitude of the wave is negligible in the limit of large Reynolds number. Assuming that wave propagation takes place in the regions $1 \ll |y - y_c| \ll R$, the vertical wavenumbers may therefore be taken as real there. The real parts of these wavenumbers, denoted by k^- and k^+, are the positive roots of the equation

$$(k^\pm)^2 = \frac{Ri(\pm\infty)}{(U(\pm\infty) - c)^2} - \alpha^2, \tag{4.9}$$

cf. (2.5).

We now consider the problem of reflection and transmission of an incident wave propagating in the positive y-direction. For reasons that have already been mentioned, the reflection and transmission coefficients will be measured in the regions $1 \ll |y-y_c| \ll R$. When the fluid is inviscid, the inviscid solution (3.1) should be proportional to $\exp(ik^+y)$ in the homogeneous region $y-y_c \gg 1$ because it should represent the transmitted inviscid wave there.

For a viscous fluid it has been shown that (4.3) is uniformly valid on every closed interval outside the critical level. For large R the solution of (2.3) may then be replaced by the outer solution (3.21) on these intervals. In the region $1 \ll y-y_c \ll R$ the outer solution should then also be proportional to $\exp(ik^+y)$ because it should represent the transmitted viscous wave in this region. When the amplitudes of the transmitted inviscid and viscous waves are normalized to unity (by adjusting the amplitude of the incident wave), it is found that the inviscid and outer solutions are identical for all $y > y_c$ because both are solutions of the same equation (2.5). Hence $A = A^+$, $B = B^+$. In view of (3.19) the inviscid and outer solutions are identical for all y. Then (4.3) implies that on every closed interval outside the critical level the solution of (2.3) is identical to the inviscid solution, at least to the approximation that $R \to \infty$.

In conclusion it is observed that in the limit as $R \to \infty$ the reflection and transmission coefficients for a wave incident on a viscous fluid are the same as in the inviscid case. When the region $y-y_c \gg 1$ is opaque, in which case there is no transmitted wave, the reflection coefficients for a viscous and an inviscid fluid remain of course the same.

These results, which are valid for all J (where J is the Richardson number at the critical level as defined by (2.2)), agree with the numerical results derived by Hazel (1967). This author, however, only treated the case $J > \frac{1}{4}$ and assumed a linear shear flow profile and a constant Brunt–Väisälä frequency. Moreover, only the transmission coefficient was considered.

Bowman, Thomas & Thomas (1980) also solved the problem numerically (for $J > 1$) and they also assumed a linear shear flow profile and a constant Brunt–Väisälä frequency. They suggested that 'the wave-amplitude attenuation factor predicted by the inviscid model is approached asymptotically in the limit of vanishingly small viscosity and thermal conductivity coefficients'. This is precisely what has been shown in this paper.

We have shown that the reflection and transmission coefficients for the inviscid case are good approximations to the viscous reflection and transmission coefficients if the Reynolds number is large. It is therefore of interest to notice that for the configuration

$$U(y) = \tfrac{1}{2}(1+\tanh y), \quad N(y) = \text{const}, \quad c = \tfrac{1}{2}, \tag{4.10}$$

the reflection and transmission coefficients for a wave incident on an inviscid fluid can be calculated explicitly because they can be expressed in terms of Γ-functions (Van Duin & Kelder 1982).

4.3. *Behaviour of the velocity components at and near the critical level*

For internal gravity waves in an inviscid, incompressible fluid the perturbed vertical velocity vanishes at the critical level. The perturbed horizontal velocity tends to infinity as the critical level is approached. This follows from (3.1)–(3.2) and from the assumption of incompressibility. This assumption implies that the perturbed horizontal velocity ψ and the vertical velocity ϕ are related according to

$$i\alpha\psi + D\phi = 0. \tag{4.11}$$

We consider next the effect of inclusion of viscosity and heat conduction on the velocity components. In that case the (perturbed) vertical velocity at the critical level vanishes as $R \to \infty$, because (4.1) implies that in this limit

$$\phi(y_c) \sim \begin{cases} \dfrac{A^+\Gamma(\frac{3}{2}+\mu)\,v_3(0)}{2\pi i}(i\alpha R U_c')^{-\frac{1}{3}(\mu+\frac{1}{2})} & (J<\frac{1}{4}, B^+=0), \\ \dfrac{B^+\Gamma(\frac{3}{2}-\mu)\,u_3(0)}{2\pi i}(i\alpha R U_c')^{\frac{1}{3}(\mu-\frac{1}{2})} & (J<\frac{1}{4}, B^+ \neq 0), \\ \dfrac{A^+\Gamma(\frac{3}{2}+i\sigma)\,v_3(0)}{2\pi i}(i\alpha R U_c')^{-\frac{1}{3}(i\sigma+\frac{1}{2})} & \\ +\dfrac{B^+\Gamma(\frac{3}{2}-i\sigma)\,u_3(0)}{2\pi i}(i\alpha R U_c')^{\frac{1}{3}(i\sigma-\frac{1}{2})} & (J>\frac{1}{4}), \end{cases} \tag{4.12}$$

where μ and σ are given by (3.11).

In the derivation of (4.12) it is assumed that away from the critical level (outside the boundary layer) the amplitude of the vertical velocity is of order unity. In that case the coefficients A^+ and B^+ in (4.12) are of the same order of magnitude. The asymptotic behaviour for $J = \frac{1}{4}$ is obtained by taking $\mu = 0$ and $\sigma = 0$ in (4.12).

For $J \geqslant \frac{1}{4}$ the rate of vanishing of the vertical velocity at the critical level is independent of J. For $J < \frac{1}{4}$ one has to distinguish between the cases $B^+ = 0$ and $B^+ \neq 0$. For $B^+ = 0$ the rate of vanishing is faster according as J is *smaller*. For $B^+ \neq 0$, on the other hand, this rate is faster according as J is *larger*.

Making use of (4.1) and (4.11), the behaviour as $R \to \infty$ of the perturbed horizontal velocity at the critical level is obtained by multiplying the right-hand side of (4.12) by $(i/\alpha)(i\alpha R U_c')^{\frac{1}{3}}$ and replacing $u_3(0)$ and $v_3(0)$ by $u_3'(0)$ and $v_3'(0)$, respectively, where (applied to u_3 and v_3 only) the prime denotes differentiation with respect to η.

It is found that the perturbed horizontal velocity at the critical level tends to infinity as $R \to \infty$. For $J \geqslant \frac{1}{4}$, the rate at which this velocity (at the critical level) tends to infinity is independent of J. For $J < \frac{1}{4}$ one has again to distinguish between the cases $B^+ = 0$ and $B^+ \neq 0$.

The coefficients $u_3(0)$ and $v_3(0)$ in (4.12) depend on P and J only. The same applies to the coefficients $u_3'(0)$ and $v_3'(0)$ in the asymptotic expression for the horizontal velocity. We will now derive closed-form expressions for $u_3(0)$, $v_3(0)$ and $u_3^{(n)}(0)$, $v_3^{(n)}(0)$, where n denotes the number of differentiations with respect to η. In this manner the Taylor expansions for $u_3(\eta)$ and $v_3(\eta)$ are also determined. We consider the coefficient $u_3(0)$ first.

Making use of the relation

$$\int_{C_3} s^{\mu-\frac{3}{2}}\,ds = 0,$$

this coefficient can be written as (cf. (3.9)),

$$u_3(0) = \int_{C_3} \{e^{-\frac{1}{3}s^3} f(s) - 1\}\, s^{\mu-\frac{3}{2}}\,ds,$$

or, alternatively,

$$u_3(0) = 2i \sin\pi(\mu-\tfrac{1}{2})\, e^{\frac{1}{3}\pi i(\mu-\frac{1}{2})} \int_0^\infty \{e^{-\frac{1}{3}t^3} f(t) - 1\}\, t^{\mu-\frac{3}{2}}\,dt \tag{4.13}$$

$$= 2i \sin\pi(\mu-\tfrac{1}{2})\, e^{\frac{1}{3}\pi i(\mu-\frac{1}{2})}\, I,$$

where

$$f(t) = {}_1F_1\left(\tfrac{1}{3}\mu+\tfrac{1}{6};\tfrac{2}{3}\mu+1;\frac{1}{3}\left(\frac{P-1}{P}\right)t^3\right).$$

Through an integration by parts we obtain

$$I = \frac{1}{\mu-\frac{1}{2}} \int_0^\infty \mathrm{e}^{-\frac{1}{3}t^3} \left\{ t^{\mu+\frac{3}{2}} f(t) - t^{\mu-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} f(t) \right\} \mathrm{d}t. \tag{4.14}$$

Making use of the relation (Erdélyi *et al.* 1953 §6.4),

$$\frac{\mathrm{d}}{\mathrm{d}x}\, {}_1F_1(p\,;q\,;x) = \frac{p}{q}\, {}_1F_1(p+1\,;q+1\,;x),$$

and introducing the new variable $\tilde{t} = t^3$, the expression for $u_3(0)$ takes the form (Erdélyi *et al.* 1953 §6.10),

$$\begin{aligned} u_3(0) = {} & 2\mathrm{i}\sin\pi(\mu-\tfrac{1}{2})\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}(\mu-\frac{1}{2})}\, \Gamma(\tfrac{1}{3}\mu-\tfrac{1}{6})\, 3^{\frac{1}{3}(\mu-\frac{7}{2})} \\ & \times \left[F\left(\tfrac{1}{3}\mu+\tfrac{1}{6}, \tfrac{1}{3}\mu+\tfrac{5}{6}; \tfrac{2}{3}\mu+1\,; \frac{P-1}{P}\right) - \frac{\mu+\frac{1}{2}}{2\mu+3}\left(\frac{P-1}{P}\right) \right. \\ & \left. \times F\left(\tfrac{1}{3}\mu+\tfrac{7}{6}, \tfrac{1}{3}\mu+\tfrac{5}{6}; \tfrac{2}{3}\mu+2\,; \frac{P-1}{P}\right) \right], \end{aligned} \tag{4.15a}$$

valid for $P > \frac{1}{2}$, and

$$\begin{aligned} u_3(0) = {} & \tfrac{2}{9}\mathrm{i}\sin\pi(\mu-\tfrac{1}{2})\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}(\mu-\frac{1}{2})}\, \Gamma(\tfrac{1}{3}\mu-\tfrac{1}{6})\, (3P)^{\frac{1}{3}(\mu+\frac{5}{2})} \\ & \times \left[F(\tfrac{1}{3}\mu+\tfrac{5}{6}, \tfrac{1}{3}\mu+\tfrac{5}{6}; \tfrac{2}{3}\mu+1\,; 1-P) - \frac{\mu+\frac{1}{2}}{2\mu+3}\left(\frac{P-1}{P}\right) \right. \\ & \left. \times F(\tfrac{1}{3}\mu+\tfrac{5}{6}, \tfrac{1}{3}\mu+\tfrac{5}{6}; \tfrac{2}{3}\mu+2\,; 1-P) \right], \end{aligned} \tag{4.15b}$$

valid for $P < 2$. The parameter μ is given by (3.11); the notation F stands for the hypergeometric function. The expression for $v_3(0)$ is obtained by replacing μ by $-\mu$ in (4.15).

The expression for the nth derivative $u_3^{(n)}(0)$ can be derived in a similar manner from the integral

$$u_3^{(n)}(0) = \int_{C_3} \mathrm{e}^{-\frac{1}{3}s^3} f(s)\, s^{n+\mu-\frac{3}{2}}\, \mathrm{d}s.$$

The result reads

$$\begin{aligned} u_3^{(n)}(0) = {} & 2\mathrm{i}\sin\pi(\mu+n-\tfrac{1}{2})\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}(\mu+n-\frac{1}{2})}\, \Gamma\left(\frac{\mu+n-\frac{1}{2}}{3}\right) 3^{\frac{1}{3}(\mu+n-\frac{7}{2})} \\ & \times F\left(\tfrac{1}{3}\mu+\tfrac{1}{6}, \tfrac{1}{3}(\mu+n-\tfrac{1}{2}); \tfrac{2}{3}\mu+1\,; \frac{P-1}{P}\right) \quad n \geqslant 1, \end{aligned} \tag{4.16a}$$

valid for $P > \frac{1}{2}$, and

$$\begin{aligned} u_3^{(n)}(0) = {} & \tfrac{2}{3}\mathrm{i}\sin\pi(\mu+n-\tfrac{1}{2})\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}(\mu+n-\frac{1}{2})}\, \Gamma\left(\frac{\mu+n-\frac{1}{2}}{3}\right) (3P)^{\frac{1}{3}(\mu+n-\frac{1}{2})} \\ & \times F(\tfrac{1}{3}\mu+\tfrac{5}{6}, \tfrac{1}{3}(\mu+n-\tfrac{1}{2}); \tfrac{2}{3}\mu+1\,; 1-P) \quad n \geqslant 1, \end{aligned} \tag{4.16b}$$

valid for $P < 2$. The expression for $v_3^{(n)}(0)$ is obtained by changing the sign of μ in (4.16). From (4.15) and (4.16) the Taylor expansions of $u_3(\eta)$ and $v_3(\eta)$ around $\eta = 0$ can now be determined.

It remains to determine the coefficients A^+ and B^+ in the asymptotic expressions for the perturbed velocity components at the critical level. These coefficients also appear in the constants α_3 and β_3 in the inner solution (3.17), cf. (3.20). Consequently, when A^+ and B^+ are known, a careful analysis of the behaviour of the perturbed vertical velocity in the viscous layer near the critical level is allowed because the inner solution (3.17), which governs this behaviour, is then completely determined by (4.15) and (4.16).

When we consider a reflection and transmission problem, A^+ and B^+ are proportional to the amplitude of the incident wave. For the configuration (4.10), A^+ and B^+ are known: they can be expressed in terms of Γ-functions because (2.5) can be reduced to the hypergeometric equation in this case (van Duin & Kelder 1982). When the incident wave propagates in the positive y-direction, and the amplitude of this wave is normalized to unity, they are given by

$$A^+ = \mathrm{e}^{-\gamma(2\pi+\mathrm{i}\ln 4)} \frac{\Gamma(\frac{1}{2}-\mu)\,\Gamma(\frac{1}{2}+\mu)\,\Gamma(-\mu)}{\Gamma(-2\mathrm{i}\gamma)\,\Gamma(\frac{5}{4}+\mathrm{i}\gamma-\frac{1}{2}\mu)\,\Gamma(-\frac{1}{4}+\mathrm{i}\gamma-\frac{1}{2}\mu)}, \tag{4.17a}$$

$$B^+ = \mathrm{e}^{-\gamma(2\pi+\mathrm{i}\ln 4)} \frac{\Gamma(\frac{1}{2}-\mu)\,\Gamma(\frac{1}{2}+\mu)\,\Gamma(\mu)}{\Gamma(-2\mathrm{i}\gamma)\,\Gamma(\frac{5}{4}+\mathrm{i}\gamma+\frac{1}{2}\mu)\,\Gamma(-\frac{1}{4}+\mathrm{i}\gamma+\frac{1}{2}\mu)}, \tag{4.17b}$$

where

$$\gamma = \tfrac{1}{2}(J-\alpha^2)^{\frac{1}{2}} \quad J > \alpha^2. \tag{4.18}$$

For $J > \frac{1}{4}$ the expressions for the absolute values of A^+ and B^+ can be reduced to

$$|A^+|^2 = \frac{\gamma}{\sigma}\mathrm{e}^{-4\pi\gamma}\frac{\sinh 2\pi\gamma}{\cosh^2\pi\sigma\,\sinh\pi\sigma}[\cosh^2\pi(\gamma-\tfrac{1}{2}\sigma)+\sinh^2\pi(\gamma-\tfrac{1}{2}\sigma)], \tag{4.19a}$$

$$|B^+|^2 = \frac{\gamma}{\sigma}\mathrm{e}^{-4\pi\gamma}\frac{\sinh 2\pi\gamma}{\cosh^2\pi\sigma\,\sinh\pi\sigma}[\cosh^2\pi(\gamma+\tfrac{1}{2}\sigma)+\sinh^2\pi(\gamma+\tfrac{1}{2}\sigma)]. \tag{4.19b}$$

5. Discussion

In the limit of large Reynolds number the thickness of the viscous layer near the critical level becomes so small that its effect on the global structure of the wave becomes negligible, with the exception, however, that in the viscous layer the actual form of the wave is radically altered.

The reflection and transmission coefficients for a viscous fluid with a critical level are, at least to the approximation that the Reynolds number tends to infinity, the same as those for an inviscid fluid with a critical level. Hence over-reflection can also occur in a (slightly) viscous fluid provided that the Richardson number at the critical level is sufficiently small. It should be noted that this implies that the occurrence of over-reflection is *not* a result of the singularity in the inviscid Taylor–Goldstein equation. Moreover, to leading order the interaction between the wave and the shear flow is not affected by viscosity and heat conduction. In cases of overlap there is agreement with the numerical results obtained by Hazel (1967) and by Bowman *et al.* (1980).

The proof of the equality of the viscous and inviscid reflection and transmission coefficients is based on the observations that (*a*) the coefficients in the leading-order outer solution (3.5) of the governing equation are interrelated according to $A^- = A^+$ and $B^- = B^+$ and (*b*) that (4.3) is uniformly valid in regions below and above the critical level.

It has been shown that there is a common region of validity of the inner and outer solutions of the governing equation. In the common region the inner and outer solutions can be joined smoothly. It turns out that only two of the six inner solutions can be matched with the outer solution. The remaining four solutions, the so-called viscous solutions, should be discarded in the matching procedure. This is at variance with the observation of Hazel (1967) that the viscous solutions are *not* negligible near the critical level. This disagreement is due to the fact that we only made a leading-order matching.

The coefficients A^+ and B^+ in the asymptotic expressions for the perturbed velocity components at the critical level can be determined by solving the Taylor–Goldstein equation. For the configuration (4.10) these coefficients can be expressed in terms of Γ-functions.

It was assumed that the dissipation near the critical level results from molecular viscosity and heat conduction. It is tempting to assume that eddy viscosity and eddy conduction of heat can also properly be described by (2.1). In that case the whole analysis as given before can be retraced by replacing the molecular kinematic viscosity and the molecular thermal conductivity by their eddy counterparts. The eddy Prandtl number may become large (Chao & Schoeberl 1984), which may lead to scaling problems, but calculations for mesosphere parameters indicate that this is not always the case because a value of about 20 is then appropriate (Fritts 1984). As far as we know, however, a justification of the above assumption (in particular its validity near the critical level) is missing.

Recently, Lindzen & Barker (1985) have shown that a small amount of damping can actually induce over-reflection in cases where it does not occur according to the inviscid theory. This result could be checked by performing a higher-order matching. The higher-order correction to the inviscid reflection coefficient should then indicate that a small amount of viscosity and heat conduction gives rise to a larger reflection coefficient. To analyse this problem in detail is beyond the scope of the present study.

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